Evidence Theory and Bayesian Probability for Characterizing Epistemic Uncertainty and Experimental Comparison of These Methods

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Problem Definition

- Test problem for methods for modeling epistemic uncertainty
- Quantify uncertainty in a function when there is imprecise information about input variables
- Function of variables $Y=Y(A, B)$
- Experts estimate intervals for A and B
- Variables A and B are independent -- our belief about A does change if we learn something about B
- What can we tell about Y?
- How good are the methods used?

Outline

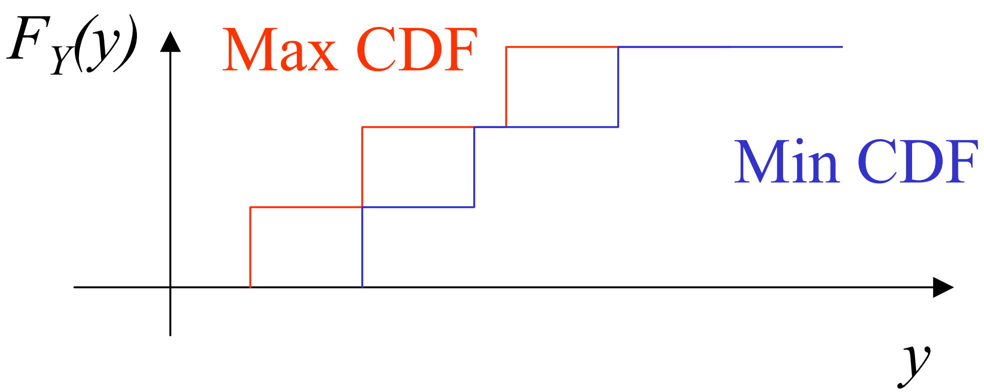
- Minimum-maximum probability method
- Bayesian method
- Experimental comparison

Min-max Probability Method for Quantifying Uncertainty in a Function

Marginal minimum and maximum probability density and cumulative distributions of input variables consistent with evidence

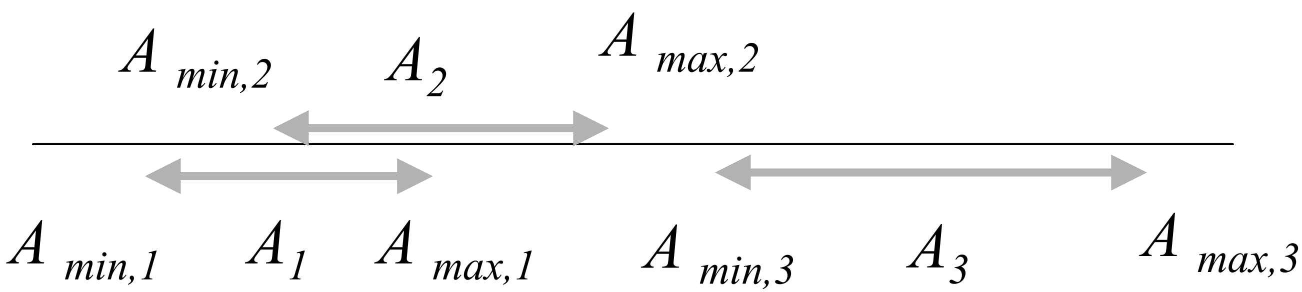
Joint minimum and maximum probability density and cumulative distributions of input variables

Minimum and maximum cumulative distribution functions (CDF) of dependent variable



Finding Maximum and Minimum Probabilities

- Evidence from experts about independent variable *A*:



- Evidence: imprecise outcomes of experiment
- Randomness and imprecision
  - Randomness: each expert gives a different answer about *A* -- we do not know who is right
  - Imprecision: experts provide intervals -- we do not know precise value of *A*

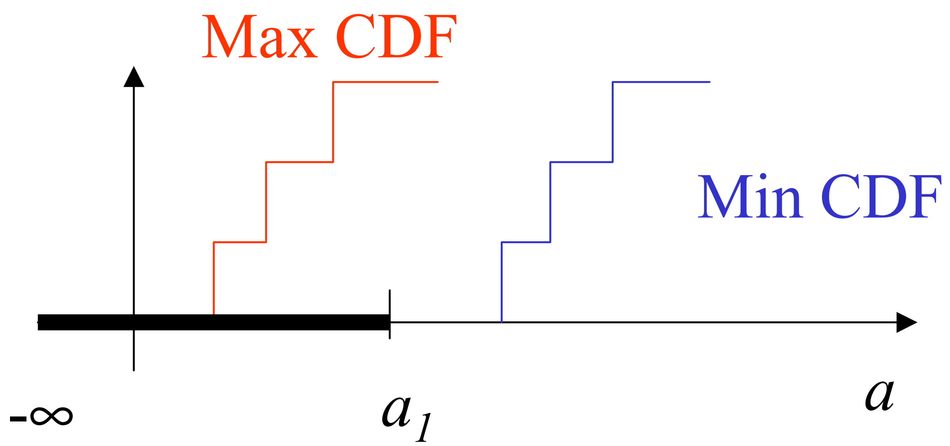
Finding Minimum and Maximum Probabilities of Events

- Optimization problem formulation for finding minimum (maximum) probability of an event, *E*
- Find $a_1, \ldots a_n$
- To minimize (maximize)

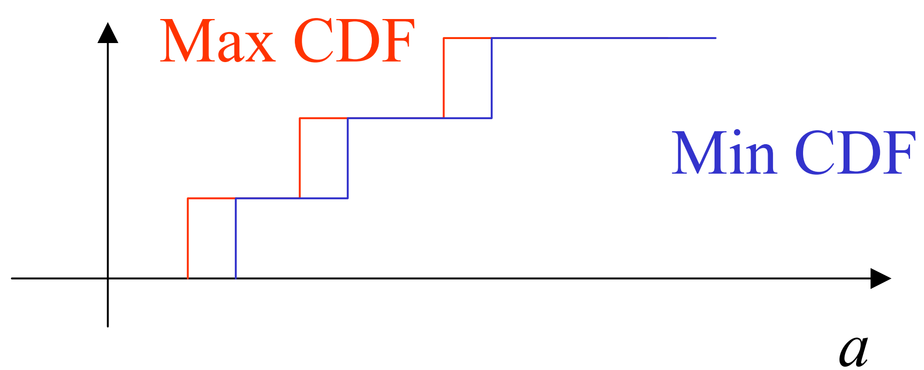
$$P(E) = \sum_{i=1}^{n} I_i \quad where\ I_i = \begin{cases} 1 & \text{if } A_i \text{ implies } E \\ 0 & \text{otherwise} \end{cases}$$

- So that a_i in $[A_{min,i}, A_{max,i}]$

Modeling Uncertainty Using Minimum and Maximum Cumulative Probability Distribution Functions (CDF)



High imprecision, low variability



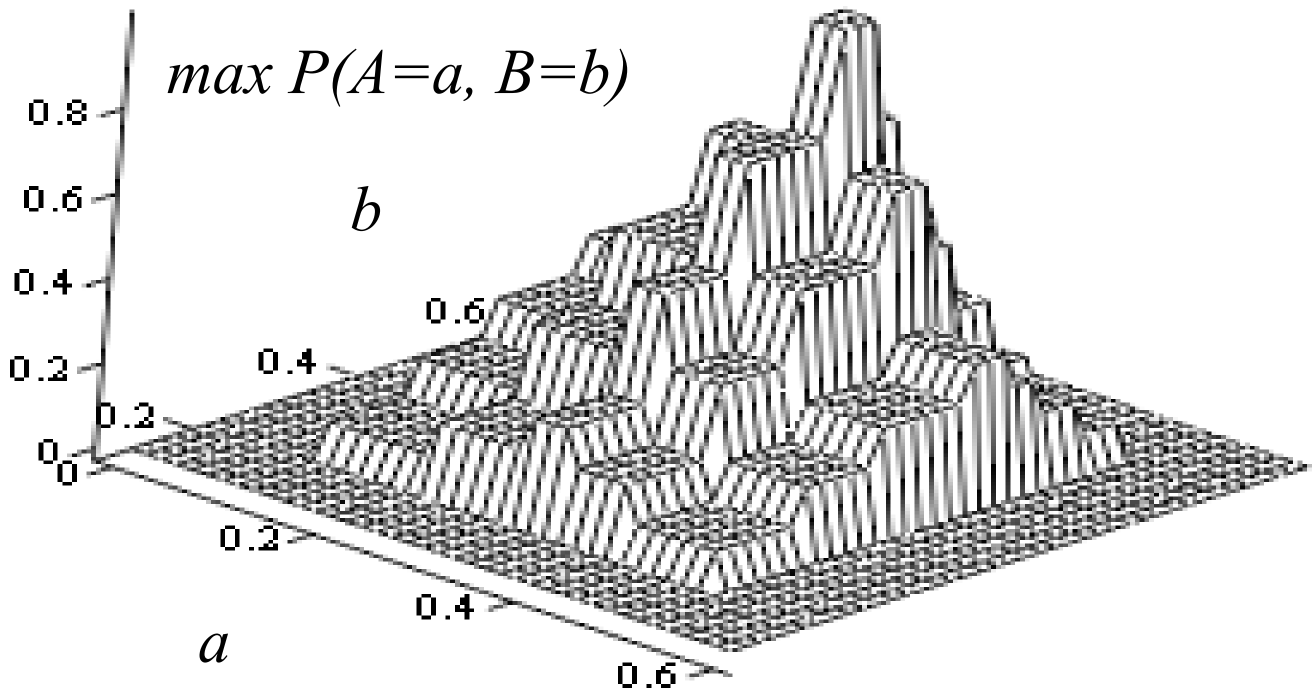
Low imprecision, high variability

Min CDF(a_1) = Min $P(A\in([-\infty, a_1]) = Belief([-\infty, a_1])$

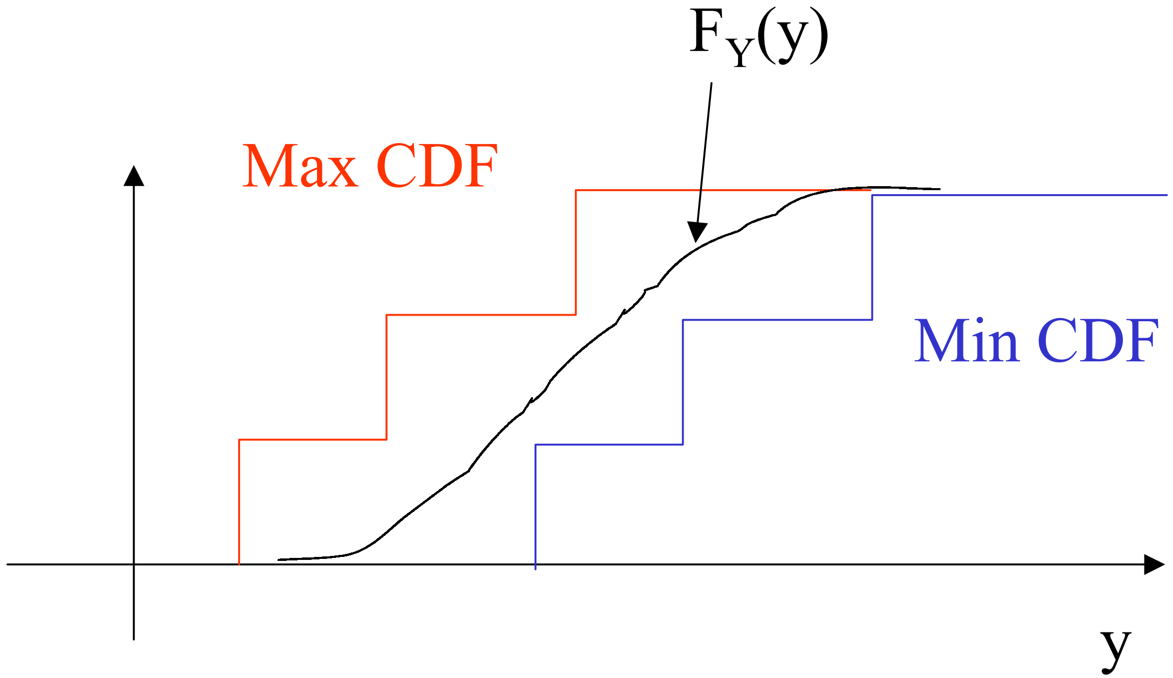
Max CDF(a_1) = Max $P(A\in([-\infty, a_1]) = Plausibility([-\infty, a_1])$

Joint Minimum and Maximum Probabilities of *A* and *B*

$$\frac{max\, P(\, A = a_1 , B = b\,)}{max\, P(\, A = a_2 , B = b\,)} = \text{independent of } b \text{ if } \max P(b) \neq 0$$

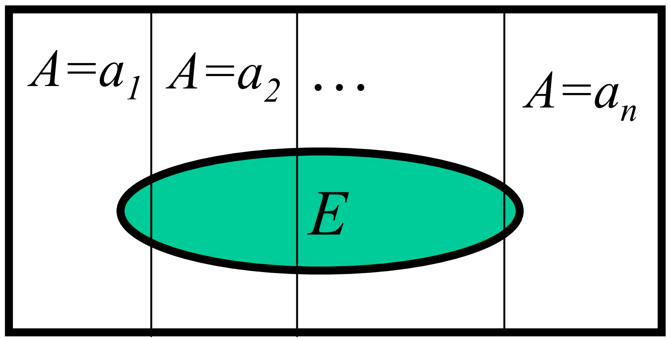


Find Min CDF and Max CDF of *Y*



Bayesian Method

- Discrete variable, *A*, evidence *E*



$$P(A=a_i\,/\,E)=\frac{P(E\,/\,A=a_i)P_0(A=a_i)}{P(E)}$$

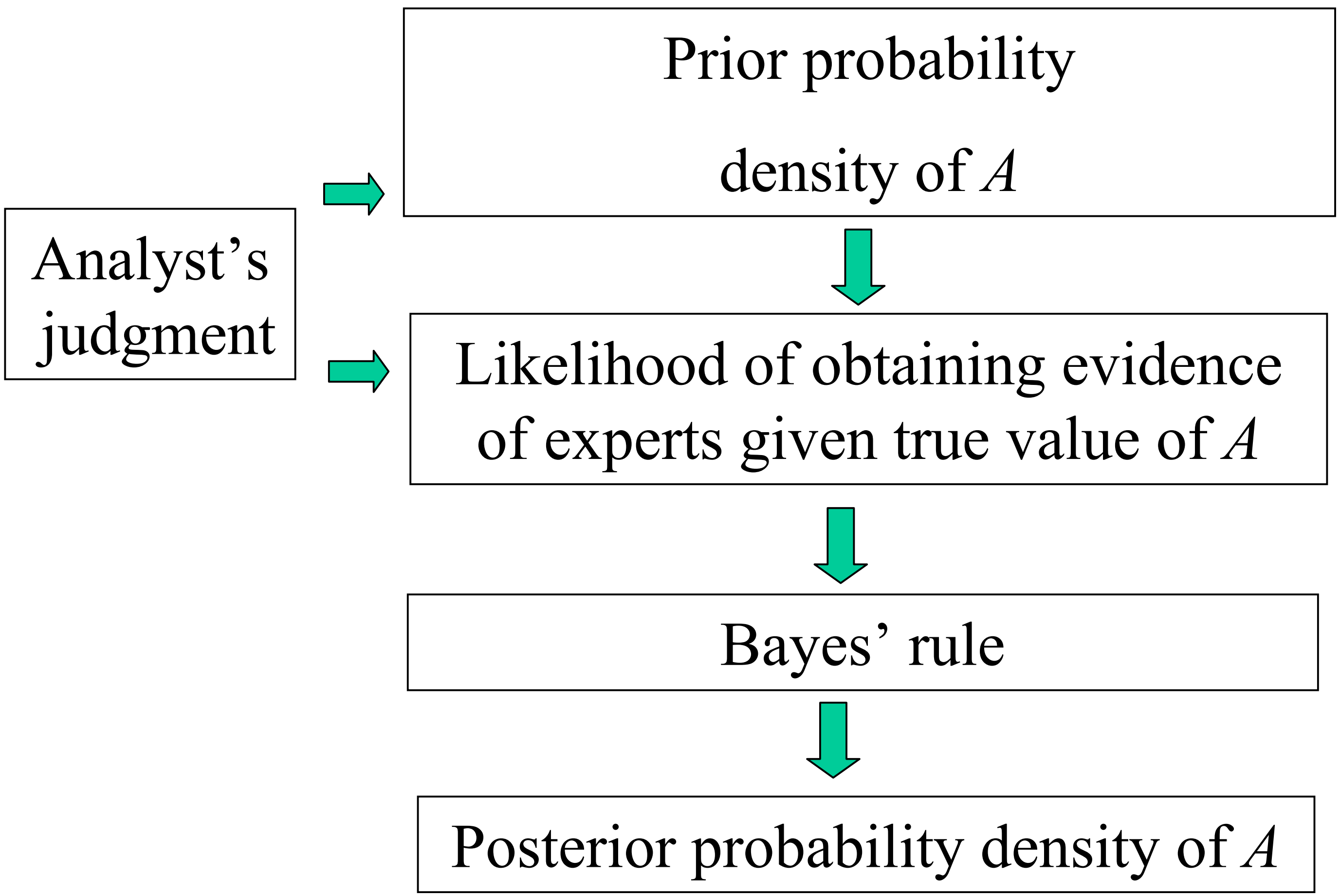
- Continuous variable, *A*, evidence *E*

Posterior state of knoweledge of A given evidence E =

c × Likelihood of evidence E given value of A × prior knowledge of A

$$f_A(a)=cL(E\,/\,A=a)f_A^0(a)$$

Bayesian Method
for Modeling Uncertainty in Variable *A*



Assumptions for Modeling Uncertainty in Independent Variables *A* and *B*

- Maximum entropy principle; uniform prior
- Evidence; a point estimate, a_i, is assumed (e.g. Interval midpoint)
- Point estimate of expert = true value + error:

$$A_i = a + e_i$$

- Errors of experts, jointly normal random variables
- Mean values of errors, estimated based on judgment
  - Unbiased expert, mean value of error = 0
- Standard deviation of error, σ_{E_i}, estimated based on judgment. Example, standard deviation=interval width/6)
- Correlation coefficients estimated based on judgment

$\rho = 0$ uncorrelated experts

$\rho > 0$ positively correlated experts

$\rho < 0$ negatively correlated experts

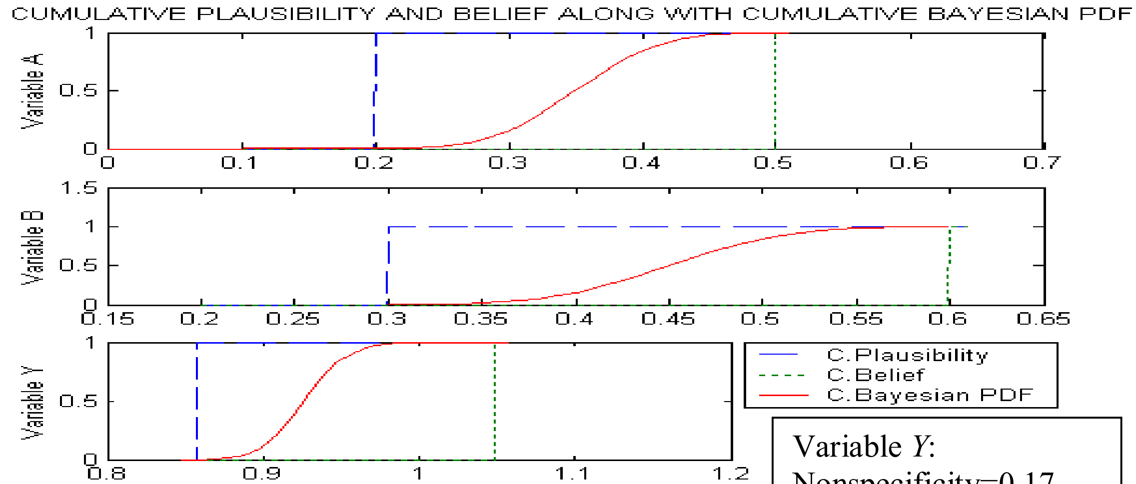
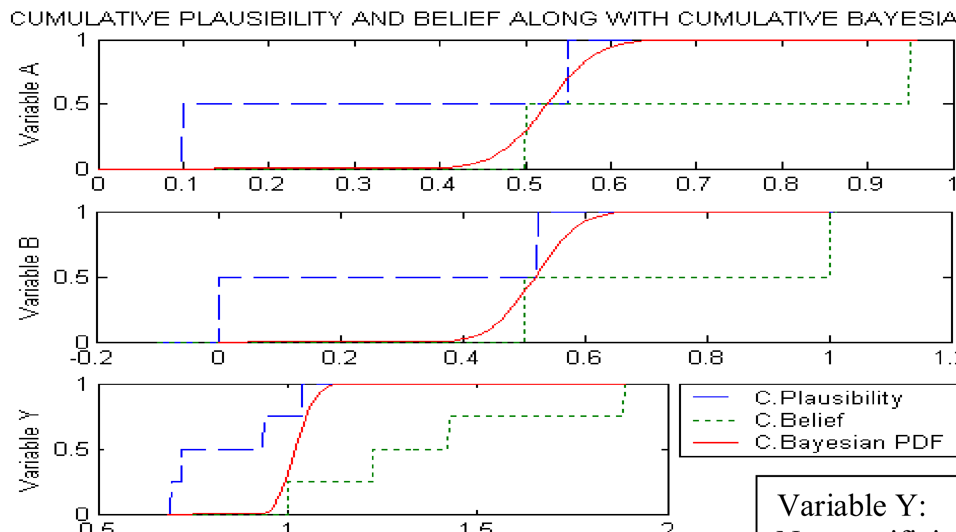
Likelihood Function

For n experts $L(\mathbf{E} \mid a) = f_{E_1 \ldots E_n}(a_1 - a, \ldots, a_n - a)$

$$= \frac{1}{(2\cdot\pi)^{\frac{n}{2}} \cdot C^{\frac{1}{2}}} . e^{-\frac{1}{2} . (a_1 - a - \overline{E}_1, \ldots, a_n - a - \overline{E}_n) \cdot C^{-1} \cdot \begin{pmatrix} a_1 - a - \overline{E}_1 \\ \vdots \\ a_n - a - \overline{E}_n \end{pmatrix}}$$

where $\mathbf{C}$ = covariance matrix of errors of experts

$$\mathbf{C} = \begin{bmatrix} \sigma_{E_1}{}^2 & \rho_{1,2}\sigma_{E_1}\sigma_{E_2} & \cdots \\ \vdots & \ddots & \vdots \\ \rho_{n,1}\sigma_{E_n}\sigma_{E_1} & \cdots & \sigma_{E_n}{}^2 \end{bmatrix}$$

Evidence	Assumptions	Results
A [0.2, 0.5] B [0.3, 0.6]	Unbiased experts error=interval width/6 (Bayesian method)	 Variable *Y*: Nonspecificity=0.17 Strife=0
A [0.1, 0.5], [0.55, 0.95] B [0, 0.5], [0.52, 1]	Independent unbiased experts error=interval with/6 (Bayesian method)	 Variable Y: Nonspecificity=0.417 Strife=0.854

Conclusions

- Max-min probability
  - Provides lower and upper bounds of probabilities
  - Treats imprecision and randomness separately, provides two measures of these uncertainties: *Nonspecificity* and *Strife*
  - Maximum (minimum) probabilities of an event identical with *Plausibility* (*Belief*) in evidence theory if $[A_{min,i}, A_{max,i}]$ are focal elements with basic probability $1/n$
  - Computation of maximum and minimum joint probabilities of A and B using optimization produces same results as evidence theory with Dempster's rule of combination
- Bayesian method
  - Accounts for correlation, bias, and credibility of experts
  - Strong axiomatic foundation
  - Bayes' rule does not directly apply when evidence consists of intervals
  - Requires analyst to make assumptions about:
    - Prior knowledge of probability distributions of random variables
    - Joint probability distributions of errors of experts
  - Does not distinguish between imprecision and randomness

Experimental Comparison

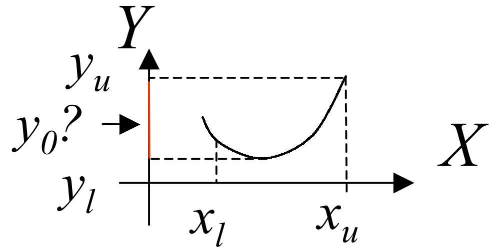
- A method for quantifying uncertainty will eventually be used for making decisions
- A good method should lead to decisions that produce desirable outcomes in the long run
- Testing can help us discover weaknesses in methods
- Test methods in terms their effectiveness for making decisions using simulations

Motivation for Selection of Test Problem

- We want to test which of two players (John and Linda) can better estimate the weight of a piece of cake
- Players take turns splitting cakes using the following rules:
  – One player cuts cake into two
  – Other player selects a part
  – Player who split cake gets remaining part
- The player whose pieces have higher total weight wins

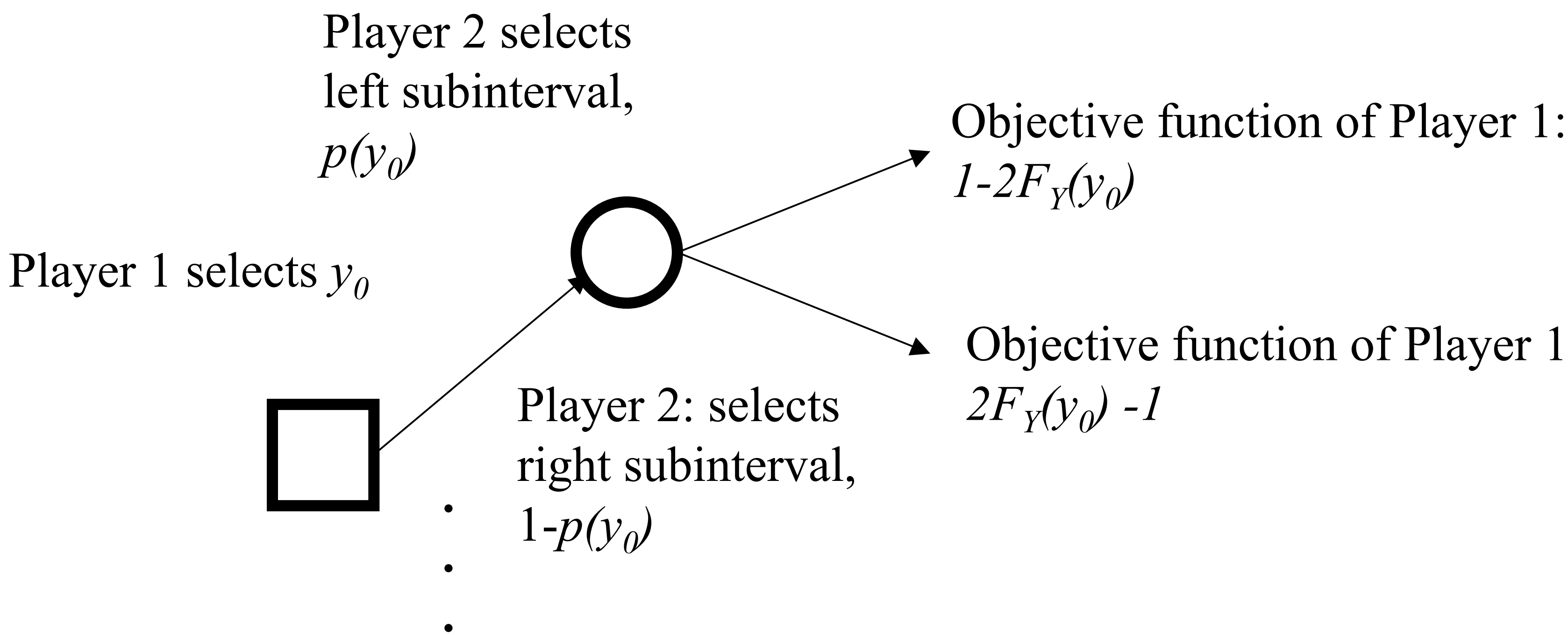
Testing Methods for Modeling Uncertainty Using Interval Splitting Problem

- We want to test which method can better estimate the probability distribution of function $Y = Y(X)$
- $X \in [x_l, x_u] \Rightarrow Y \in [y_l, y_u]$
- Evidence about X from experts
- Two players;
  - Each player wants to get subinterval of Y with higher probability
  - Each player uses different method for modeling uncertainty



- Players take turns splitting interval of Y using the following rules:
  - One player splits interval into two
  - Other player selects a subinterval
  - Player who split interval gets remaining subinterval
- The player whose pieces have higher average probability wins

Decision Tree



Deciding Who Made Better Decisions